Dr. Lafta Algaisy — Engineering & Numerical Analysis

Matrices :-

Def :- A matrix is a set of numbers arranged as rectangular arrays.

$$EX \quad \begin{array}{cl} \begin{matrix} & & \text{Column} \\ & & \downarrow \end{matrix} & \\ \begin{bmatrix} 2 & 3 & 1 \\ -5 & 5 & 5 \end{bmatrix} & \leftarrow \text{row} \quad (2\times3) \end{array}$$

In general

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

m : number of row.
n : number of Column.

Note

1) if $n = m$ then the matrix is called square matrix.
2) Sometimes a matrix A is denoted by $A = a_{\underset{\text{row}}{i}\,\overset{\text{column}}{j}}$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -1 & 5 \\ 1 & 3 & 4 \end{bmatrix}, \quad a_{23} = 5, \; a_{31} = 1, \; a_{22} = -1, \; a_{33} = 4.$$

Def :- Zero matrix $A = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & \cdots & 0 \end{bmatrix}$

Addition and Subtruction:-

1) the number of column in the 1^{st} matrix and 2^{nd} must be equally.

2) The number of row in the 1^{st} matrix & 2^{nd} matrics must be equally.

EX

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 0 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 4 \\ 5 & 6 & 0 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 8 \\ 8 & 8 & 1 \\ 2 & 3 & 9 \end{bmatrix}$$

EX

$$\begin{bmatrix} 1 & 5 & 3 \\ 2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & -1 & 5 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 8 \\ 4 & 4 & 5 \end{bmatrix}$$

Multiplication :-

Note:

1) the number of column in the 1^{st} matrix must be equal to row in the 2^{nd} matrix.

2) if $AB = 0$ then not necessary $A = 0$ or $B = 0$

3) Identity matrix [unit matrix I_n].

$$I_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \cdots & 0 \\ & & & \ddots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Determinant :-

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is square matrix, then $\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= ad - bc$

Ex $A = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \Rightarrow \det A = |A| = \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = 3 \times 0 - 2 \times 1 = -2$

1st method :- if the determinant of order 3.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ -2 & 2 & 3 \end{vmatrix} = 1 \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ -2 & 3 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ -2 & 2 \end{vmatrix}$$

$$= 1(-3-2) - 2(6+2) + 3(4-2) = -5 - 16 + 6 = -15$$

2nd method :-

$$\left|\begin{matrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ -2 & 2 & 3 \end{matrix}\right| \begin{matrix} 1 & 2 \\ 2 & -1 \\ -2 & 2 \end{matrix} = -3 - 4 + 12 - (6 + 2 + 12) = -15$$

Note :- $A I_n = I_n A = A$

Ex $A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A I_2 = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$, $I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$

Ex :- if $A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$, $A I^5 = A$

Transpose: A^T

Ex if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 5 \\ 3 & -1 & 5 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 2 & -1 \\ 3 & 5 & 5 \end{bmatrix}$

Note:- ① $(A+B)^T = A^T + B^T$

② $(AB)^T = B^T \cdot A^T$

Inverse of matrices:- $A^{-1} = \dfrac{\text{adjoint } A}{\det A}$

Where A is Square matrix $\Rightarrow$ $\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{m2} \\ A_{12} & A_{22} & \cdots & A_{m2} \\ \vdots & & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nm} \end{bmatrix}$

Ex

$\text{adj.}\, A = \begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix}$, $\text{adj } A = \begin{bmatrix} 3 & -5 \\ -4 & 1 \end{bmatrix}$

Ex if $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}$, find $\text{Adj } A$?

$\text{adj } A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 0 \\ 0 & -2 & -3 \end{bmatrix}$, $\text{adj } A = \begin{bmatrix} \begin{vmatrix} -1 & 0 \\ -2 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \end{bmatrix}$

Ex $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}$, find A^{-1}

$$A^{-1} = \frac{\text{adj } A}{\det A} = \frac{\begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & 7 \end{bmatrix}}{17}$$

note:- $AA^{-1} = A^{-1}A = I_n$

EX

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix} * \frac{1}{17}\begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant Matrix (4x4)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & -1 & 2 & 3 \\ 1 & 6 & 4 & -2 \end{bmatrix}$$

$$= 1\begin{vmatrix} 3 & 2 & 1 \\ -1 & 2 & 3 \\ 6 & 4 & -2 \end{vmatrix} - 2\begin{vmatrix} 4 & 2 & 1 \\ 0 & 2 & 3 \\ 1 & 4 & -2 \end{vmatrix} + 3\begin{vmatrix} 4 & 3 & 1 \\ 0 & -1 & 3 \\ 1 & 6 & -2 \end{vmatrix} - 4\begin{vmatrix} 4 & 3 & 2 \\ 0 & -1 & 2 \\ 1 & 6 & 4 \end{vmatrix}$$

$$= 1[3(-4-12) - 2(2-18) + 1(-4-12)] - 2[4(-4-12) + 1(6-2)] +$$
$$3[4(2-18) + (9+1)] - 4[4(-4+12) + (6+2)] = 150.$$

Determinant matrix (4x4) by General Laplace Expansion (GLE)

EX

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & -1 & 2 & 3 \\ 1 & 6 & 4 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}\cdot\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}\cdot\begin{vmatrix} -1 & 3 \\ 6 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 4 & 1 \end{vmatrix}\cdot\begin{vmatrix} -1 & 2 \\ 6 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}\cdot\begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix}$$
$$- \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}\cdot\begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}\cdot\begin{vmatrix} 0 & -1 \\ 1 & 6 \end{vmatrix}$$

$$= -5\cdot(-16) - (-10)\cdot(-16) + (-15)\cdot(-16) + (-5)\cdot(-3) + (-10)\cdot(-2)$$
$$+ (-5)\cdot 1 = 150$$

Linear Transformation :-

Vectors, matrices, and determinants find immediate application in the Solution of linear algebric equation. For instance, ~~also~~ a solution of the equation.

$$2x - y = 5 \quad \text{--- } (1)$$

if we have two equations.

$$\begin{aligned} 2x - y &= 5 \quad \text{--- } (1) \\ x - 2y &= 4 \quad \text{--- } (2) \end{aligned} \Rightarrow \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Our interest all the moment is to learn how to solve general System of linear algebric equations.

$$a_1 x_1 + a_2 x_2 + \cdots\cdots + a_n x_n = b$$

Now. Suppose we have m simultanous linear equation

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + & \cdots\cdots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + & \cdots\cdots + a_{2n} x_n = b_2 \\ \vdots \qquad & \qquad\qquad \vdots \qquad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + & \cdots\cdots\cdots a_{mn} x_n = b_m \end{aligned}$$

$x_1, x_2, \cdots, x_n$. refers to a system of linear System.

$b_1, b_2, \cdots, b_m$ are Called Constants of the System.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\therefore \; Ax = b \Rightarrow \boxed{x = A^{-1} b}$$

Ex: Solve the following set of three simultanous equation:-

$$x_1 + 3x_2 + x_3 = 1$$
$$2x_1 + x_2 = 2$$
$$x_1 + x_2 + 2x_3 = 3$$

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$Ax = b \Rightarrow x = A^{-1} b$$

$$A^{-1} = \frac{adj[A]}{\det A} = \frac{\begin{bmatrix} 2 & -5 & -1 \\ -4 & 1 & 2 \\ 1 & 2 & -5 \end{bmatrix}}{-9} \Rightarrow x = -\frac{1}{9} \begin{bmatrix} 2 & -5 & -1 \\ -4 & 1 & 2 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{9} \begin{bmatrix} -11 \\ 4 \\ -10 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/9 \\ -4/9 \\ 10/9 \end{bmatrix}.$$

Gramer's rule to find $x_1, x_2, \dots, x_n$.

<u>EX</u> Solve the following set of three simultaneous equation by "Gramer's rule":

$$x_1 + 3x_2 + x_3 = 1$$
$$2x_1 + x_2 = 2$$
$$x_1 + x_2 + 2x_3 = 3$$

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}} = 11/9, \quad x_2 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 3 & 2 \end{vmatrix}}{-9} = \frac{-4}{9}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{vmatrix}}{-9} = \frac{10}{9}$$

<u>Properties of the matrices</u>:-

1) Diagonal matrix if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad D_3 = D_g(1,5,6) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

2) A matrix A is upper triangular if and only if it is square and $a_{ij} = 0$ if $i > j$.

$$\begin{bmatrix} 1 & 6 & -4 \\ 0 & 5 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 5 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

3) A matrix A is lower triangular if and only if it is square and $a_{ij} = 0$, if $i < j$

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ -1 & 1 & 5 \end{bmatrix}$$

4) Power matrix

$$A^n = AA \cdots A^n$$

Ex $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, find A^3

$$A^3 = A \cdot A \cdot A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

5) Symmetric matrix

AA^T and A^TA are symmetric $\begin{bmatrix} 1 & 5 & -2 \\ 5 & 3 & 4 \\ -2 & 4 & 1 \end{bmatrix}$

6) For every square matrix A. $\det A^T = \det A$

7) if A and B are matrices of the same orders then $(\det A) \cdot (\det B) = \det(AB)$.

8) A square matrix whose determinant is Zero is said to be singular.

9) " " " " " is not Zero " "

" " non Singular.

10) if A and B are non singular $n \times n$ matrices, then

$$(AB)^{-1} = B^{-1}A^{-1}$$

11) $\det(AA^{-1}) = \det A \cdot \det A^{-1} = 1$

12) $A^{-n} = (A^{-1})^n$

Ex if $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, then find $[A]^{-2}$

$$[A]^{-2}=[A^{-1}]^{2} = A^{-1}=\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}/2 \ , (A^{-1})^{2}=\begin{bmatrix}5.5 & -2.5\\ -3.75 & 1.75\end{bmatrix}.$$

Eigenvalues and Eigenvectors :-

An "eigenvalue" or "characteristic value" (or latent root) of a given $n\times n$ matrix $A=[a_{jk}]$ is a real or complex number λ such that the vector equation

$$AX=\lambda X \quad \text{——} (1)$$

has a nontrivial solution, that is, a solution $X\neq 0$, which is then called an "eigenvector" or "characteristic vector" of A corresponding to the eigenvalue λ. the set of all eigenvalues of A is called the "spectrum" of A."

Equation (1) can be written

$$(A-\lambda I)X=0$$

where I is the $n\times n$ unit matrix. this homogenous system has a nontrivial solution if and only if the "characteristic determinant" $\det(\lambda I-A)$ is 0.

Indeed, eigenvalue problems come up all the time in engineering, physics, geometry, numerics, theoretical math, EX find the eigen values of matrix $A = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}$

the characteristic equation $|\lambda I - A| = 0$

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} \right| = 0$$

$$= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} \right| = 0 \Rightarrow \begin{vmatrix} \lambda-4 & 5 \\ -1 & \lambda+2 \end{vmatrix} = 0$$

$$= (\lambda-4)(\lambda+2) + 5 = 0$$

$$= \lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda-3)(\lambda+1) = 0$$

$$\lambda_2 = -1, \lambda_2 = 3$$

EX Find eigen value of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$

$$|\lambda I - A| = 0 \Rightarrow \left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix} = 0$$

$$\lambda[\lambda(\lambda+6) + 11] + 6 = 0 \Rightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$$(\lambda+1)(\lambda^2 + 5\lambda + 6) = 0$$

$$(\lambda+1)(\lambda+2)(\lambda+3) = 0$$

$$\lambda = -1, \lambda_2 = -2, \lambda_3 = -3$$

$$\begin{array}{r} \lambda^2 + 5\lambda + 6 \\ \lambda+1 \overline{\big) \lambda^3 + 6\lambda^2 + 11\lambda + 6} \\ \lambda^3 + \lambda^2 \qquad\qquad \\ \hline 5\lambda^2 + 11\lambda \quad \\ 5\lambda^2 + 5\lambda \quad \\ \hline 6\lambda + 6 \\ 6\lambda + 6 \\ \hline 0 \quad 0 \end{array}$$

Eigenvector :- the eigenvector is the value of $x_1, x_2, \dots, x_n$ at each eigen value. $\lambda_1, \lambda_2, \dots, \lambda_n$.

① where eigen value is distinct value -

$$[\lambda I - A][X] = 0$$

Ex find eigen value and eigenvector for $A = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}$

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda-4 & 5 \\ -1 & \lambda+2 \end{vmatrix} = 0 \Rightarrow (\lambda-4)(\lambda+2) + 5 = 0$$

$\lambda_1 = -1, \quad \lambda_2 = 3$

at $\lambda_1 = -1$

$$\begin{bmatrix} -5 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$-5x_1 + 5x_2 = 0$

$-x_1 + x_2 = 0 \Rightarrow x_2 = x_1$ -- ①

let $x_1 = 1 \Rightarrow x_2 = 1$

$\therefore P_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

at $\lambda_2 = 3$

$$\begin{bmatrix} -1 & 5 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$-x_1 + 5x_2 = 0$

$-x_1 + 5x_2 = 0 \Rightarrow x_2 = \frac{1}{5} x_1$

$x_1 = 1 \Rightarrow x_2 = \frac{1}{5}$

$\therefore P_2 = \begin{bmatrix} 1 \\ 1/5 \end{bmatrix}$

$\therefore$ eigenvector $= P = \begin{bmatrix} 1 & 1 \\ 1 & 1/5 \end{bmatrix}$

According to eq(1), we can prove our results.

$$AX = \lambda X$$

at $\lambda_1 = -1$ and $P_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad -1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

at $\lambda_2 = 3$ and $P_2 = \begin{bmatrix} 1 \\ 1/5 \end{bmatrix}$

$$\begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3/5 \end{bmatrix}, \quad 3\begin{bmatrix} 1 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3/5 \end{bmatrix}$$

EX Find eigenvector for $A = \begin{bmatrix} 1 & -2 & 4 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 2 & -4 \\ 1 & \lambda+1 & -2 \\ -1 & -1 & \lambda-1 \end{vmatrix}$$

$$(\lambda-1)[(\lambda+1)(\lambda-1)-2] - 2[\lambda-1-2] - 4[-1+\lambda+1] = 0$$

$$\therefore \lambda^3 - \lambda^2 - 9\lambda + 9 = 0$$

$$(\lambda-1)(\lambda^2-9) = 0 \Rightarrow (\lambda-1)(\lambda-3)(\lambda+3) = 0$$

at $\lambda_1 = -3 \Rightarrow \begin{bmatrix} -4 & 2 & -4 \\ 1 & -2 & -2 \\ -1 & -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$$-4x_1 + 2x_2 - 4x_3 = 0 \quad \text{(1)}$$
$$x_1 - 2x_2 - 2x_3 = 0 \quad \text{(2)} \quad \text{multiply by } 2$$
$$-x_1 - x_2 - 4x_3 = 0$$

(1) & (2)

$$-4x_1 + 2x_2 - \cancel{4x_3} = 0$$
$$2x_1 - 4x_2 - \cancel{4x_3} = 0$$

$$-6x_1 + 6x_2 = 0 \Rightarrow \boxed{x_2 = x_1}$$

① + ③

$-4x_1 + 2x_2 - 4x_3 = 0$

$-4x_1 - 4x_2 - 16x_3 = 0$ by 4

$6x_2 + 12x_3 = 0 \Rightarrow 12x_3 = -6x_2 \Rightarrow \boxed{x_3 = -\frac{1}{2}x_2}$

$\therefore P_1 = \begin{bmatrix} 1 \\ 1 \\ -1/2 \end{bmatrix} \xRightarrow{\times 2} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

at $\lambda_2 = 1$

$$\begin{bmatrix} 0 & 2 & -4 \\ 1 & 2 & -2 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0]$$

$2x_2 - 4x_3 = 0$

$x_1 + 2x_2 - 2x_3 = 0$

$-x_1 - x_2 = 0$

① + ②

$2x_2 - 4x_3 = 0$

$2x_1 + 4x_2 - 4x_3 = 0$

$-2x_1 - 2x_2 = 0 \Longrightarrow \boxed{x_2 = -x_1}$

① + ③

$2x_2 - 4x_3 = 0$

$-2x_1 - 2x_2 = 0$

$-2x_1 = 4x_3 \Rightarrow \boxed{x_3 = -\frac{1}{2}\cancel{x_1}}$

$\therefore P_2 = \begin{bmatrix} 1 \\ -1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

at $\lambda_3 = 3$

$$\begin{bmatrix} 2 & 2 & -4 \\ 1 & 4 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

$$2X_1 + 2X_2 - 4X_3 = 0$$
$$X_1 + 4X_2 - 2X_3 = 0$$
$$-X_1 - X_2 + 2X_3 = 0$$

1 & 2

$$\cancel{2X_1} + 2X_2 - \cancel{4X_3} = 0$$
$$\cancel{2X_1} + 8X_2 - \cancel{4X_3} = 0$$

$$X_2 = 0$$

1 & 3

$$\cancel{2X_1 + 2X_2 - 4X_3 = 0}$$
$$\cancel{-2X_1 - 2X_2 + 4X_3 = 0}$$

$$2X_1 - 4X_3 = 0$$
$$-X_1 + 2X_3 = 0 \ast 2 \quad -2X_1 + 4X_3 = 0$$

$$2X_1 - 4X_3 = 0 \Rightarrow 4X_3 = 2X_1 \Rightarrow X_3 = \frac{1}{2}X_1$$
$$-2X_1 + 4X_3 = 0 \Rightarrow 4X_3 = 2X_1 \Rightarrow X_3 = \frac{1}{2}X_1$$

$$X_1 = 1 \Rightarrow X_3 = \frac{1}{2}$$

$$\therefore P_3 = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix} \Rightarrow P_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 2 & 2 & 2 \\ 2 & -2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

② Multiple Eigenvalue :-

For a matrix with multiple eigen values, for example.
the first eigenvector is obtained from the equation

$$[\lambda I - A][X] = 0$$

the second eigen vector is obtained from

$$[\lambda I - A][X] = X_1$$

EX find the eigenvalue and eigenvector for $A = \begin{bmatrix} -3 & 2 \\ 0 & -3 \end{bmatrix}$

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda+3 & -2 \\ 0 & \lambda+3 \end{vmatrix} = 0$$

$$(\lambda+3)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -3$$

at $\lambda_1 = -3 \Rightarrow \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \Rightarrow -2X_2 = 0 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\therefore P_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

at $\lambda_2 = -3$

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow -2X_2 = 1 \Rightarrow X_2 = -\frac{1}{2}$$

$$\therefore P_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \Rightarrow P_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

<u>Ex</u> find eigen vector for $A = \begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$

$|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-3 & 2 & 5 \\ -4 & \lambda+1 & 5 \\ 2 & 1 & \lambda+3 \end{vmatrix} = 0$$

$\lambda_1 = -5$, $\lambda_2 = 2$, $\lambda_3 = 2$

at $\lambda_1 = -5 \Rightarrow \begin{bmatrix} -8 & 2 & 5 \\ -4 & -4 & 5 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$-8x_1 + 2x_2 + 5x_3 = 0$ —— (1) $\Rightarrow$ $x_2 = \frac{2}{3} x_1$

$-4x_1 - 4x_2 + 5x_3 = 0$ —— (2) $\quad x_3 = 2x_2$

$2x_1 + x_2 - 2x_3 = 0$ —— (3)

$\therefore P_1 = \begin{bmatrix} 1 \\ 2/3 \\ 4/3 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$

at $\lambda_2 = 2$

$$\begin{bmatrix} -1 & 2 & 5 \\ -4 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-x_1 + 2x_2 + 5x_3 = 0$ —— (1)

$-4x_1 + 3x_2 + 5x_3 = 0$ —— (2)

$2x_1 + x_2 + 5x_3 = 0$ —— (3)

$\left.\begin{array}{l} x_2 = 3x_1 \\ x_3 = -\frac{1}{3} x_2 \end{array}\right\} \Rightarrow P_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

at $\lambda_3 = 2$

$$\begin{bmatrix} -1 & 2 & 5 \\ -4 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$-x_1 + 2x_2 + 5x_3 = 1 \quad \text{——— } (1)$$

$$-4x_1 + 3x_2 + 5x_3 = 3 \quad \text{——— } (2)$$

$$2x_1 + x_2 + 5x_3 = -1 \quad \text{——— } (3)$$

$$x_2 = 3x_1 + 2$$

$$x_3 = \frac{1}{5} - \frac{1}{3} x_2$$

$$\therefore P_3 = \begin{bmatrix} 5 \\ 25 \\ -8 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 3 & 25 \\ 4 & -1 & -8 \end{bmatrix}$$

Reduction to diagonal form :-

A square matrix $A(n \times n)$ of eigenvalue $x_1, x_2, x_3, \ldots x_n$

$$AX_i = \lambda_i X_i \quad \text{——— } (1) \qquad i = 1, 2, 3, \ldots, n.$$

if $P = [X_1, X_2, \ldots X_n]$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$

$$A[X_1 X_2 \ldots X_n] = [X_1 X_2 \ldots X_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

$$AP = P\Lambda \quad \text{——— } (2)$$

Premultiplying by P^{-1} $\Rightarrow$ $P^{-1}AP = P^{-1}P\Lambda$

$$\Rightarrow P^{-1}AP = \Lambda \quad \text{—— (3)}$$

$$\Lambda^2 = P^{-1}AP \cdot P^{-1}AP = P^{-1}A^2P, \quad \Lambda^3 = P^{-1}A^3P.$$

$$\therefore \Lambda^r = P^{-1}A^rP \quad \text{—— (4)}$$

Premultiplying by P & Post multiplying by P^{-1}

$$P\Lambda^r P^{-1} = PP^{-1}A^rPP^{-1}$$

$$\therefore \boxed{A^r = P\Lambda^r P^{-1}} \quad \text{--- (5)}$$

A matrix function (A) can be expressed as

$$f(A) = a_0 I_1 + a_1 A + a_2 A^2 + \cdots + a_K A^K$$

$$f(A) = P f(\Lambda) P^{-1}$$

$$\therefore \boxed{e^{A} = P e^{\Lambda} P^{-1}}$$

EX For the matrix $A=\begin{bmatrix}2&3\\2&1\end{bmatrix}$. find A^{25} & e^{At} for the matrix A?

Sol $|\lambda I-A|=0 \Rightarrow \begin{vmatrix}\lambda-2 & -3\\ -2 & \lambda-1\end{vmatrix}=0 \Rightarrow (\lambda-1)(\lambda-2)-6=0$

$\lambda_1=-1,\ \lambda_2=4.$

at $\lambda_1=-1 \Rightarrow \begin{bmatrix}-3&-3\\-2&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=0$

$\left.\begin{array}{l}-3x_1-3x_2=0\\-2x_1-2x_2=0\end{array}\right\} \Longrightarrow x_2=-x_1 \quad \therefore P_1=\begin{bmatrix}1\\-1\end{bmatrix}$

at $\lambda_2=4$

$\begin{bmatrix}2&-3\\-2&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=0 \Rightarrow \left.\begin{array}{l}2x_1-3x_2=0\\-2x_1+3x_2=0\end{array}\right\} \Rightarrow x_2=\frac{2}{3}x_1$

$\therefore P_2=\begin{bmatrix}3\\2\end{bmatrix}$

$\therefore P=\begin{bmatrix}1&3\\-1&2\end{bmatrix}.$

$\therefore \Lambda=P^{-1}AP=\frac{1}{5}\begin{bmatrix}2&-3\\1&1\end{bmatrix}\begin{bmatrix}2&3\\2&1\end{bmatrix}\begin{bmatrix}1&3\\-1&2\end{bmatrix}$

$\Lambda=\begin{bmatrix}-1&0\\0&4\end{bmatrix}$

$\therefore A^{25}=P\Lambda^{25}P^{-1}=\frac{1}{5}\begin{bmatrix}1&3\\-1&2\end{bmatrix}\begin{bmatrix}-1^{25}&0\\0&4^{25}\end{bmatrix}\begin{bmatrix}2&-3\\1&1\end{bmatrix}$

$A^{25}=\frac{1}{5}\begin{bmatrix}-2+3*4^{25} & 3+3*4^{25}\\ 2+2*4^{25} & -3+2*4^{25}\end{bmatrix}$

$$e^{At} = P e^{\Lambda t} P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{4t} \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix} \Rightarrow$$

$$e^{At} = \frac{1}{5}\begin{bmatrix} 2e^{-t}+3e^{4t} & -3e^{-t}+3e^{4t} \\ -2e^{-t}+2e^{4t} & 3e^{-t}+2e^{4t} \end{bmatrix}.$$

Jordan Canonical form :-

In linear algebra, a Jordan normal form (often called Jordan canonical form) of a linear operator on a finite-diemensional vector space is an upper triangular matrix of a particular form called a "Jordan matrix".

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ 0 & 0 & & \cdots & & 1 \\ a_1 & a_2 & a_3 & \cdots & & a_n \end{bmatrix}$$

will be find the eigen value by.

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & & & & & \lambda - 1 \\ -a_1 & -a_2 & -a_3 & \cdots & & -a_n \end{vmatrix}$$

$$\therefore \lambda^n - a_1\lambda^{n-1} - a_2\lambda^{n-2} - \cdots\cdots a_0 = 0$$

a) if distinct eigen value - The eigen vector will be,

$$P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \lambda_1^2 & \lambda_2^2 & & \lambda_m^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^n & \lambda_2^n & & \lambda_m^n \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

b) If multiple eigenvalue,

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \lambda_1 & \frac{d\lambda_1/dt}{1!} & \frac{d^2\lambda_1/dt}{1!} & \cdots & \\ \lambda_1^2 & d\lambda_1^2/2! & \frac{d^2\lambda_1^2/dt}{2!} & \cdots & \\ \vdots & \vdots & \vdots & & \\ \lambda_1^n & & & & \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Jordan canonical form.

EX find the eigenvalue and eigen vector and Λ for $A = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}$

$$|\lambda I - A| = 0 \Rightarrow \lambda^2 + 4\lambda - 5 = 0$$

$$\lambda_1 = -5 , \lambda_2 = 1$$

$$\therefore P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix}$$

$$\therefore \Lambda = P^{-1}AP \Rightarrow \Lambda = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

EX find eigenvalue and eigenvector for $A = \begin{bmatrix} 0 & 1 \\ -25 & 10 \end{bmatrix}$

$$|\lambda I - A| = 0 \Rightarrow \lambda^2 - 10\lambda + 25 = 0$$

$$\lambda_1 = \lambda_2 = 5$$

$$P = \begin{bmatrix} 1 & 0 \\ \lambda_1 & 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \quad \therefore P^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$$

$$\Lambda = P^{-1}AP = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$$ Jordan Canonical form.

EX if $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ \lambda_1 & 1 & 0 \\ \lambda_1^2 & 2\lambda_1 & \frac{2}{2!} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\Lambda = P^{-1} A P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

the Hamilton theorem :-

The Hamilton theorem states that every square matrix satisfies its own characteristic equation. For an $n \times n$ square matrix A with a characteristic equation.

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + a_{n-1}\lambda + a_n = 0$$

the following square matrix,

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_0 I = [0]$$

Ex For the matrix $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, find A^2, A^3, A^4, and A^{-1} by Hamilton theorem.

$|\lambda I - A| = 0 \Rightarrow \lambda^2 - 5\lambda + 4 = 0$

$A^2 - 5A - 4I = 0 \Rightarrow \boxed{A^2 = 5A - 4I}$

$A^3 = A^2 A = (5A - 4I)A = 5A^2 - 4A$

$= 5(5A - 4I) - 4A$

$= 25A - 4A - 20I = \boxed{21A - 20I}$

$A^4 = A^3 A = 21A^2 - 20A = 21(5A - 4I) - 20A$

$= \boxed{85A - 84I}$

for A^{-1}

$A^2 - 5A + 4I = [0]$

$A(A - 5I + 4A^{-1}) = 0 \Rightarrow 4A^{-1} = 5I - A$

$A^{-1} = \frac{5I - A}{4} \Rightarrow A^{-1} = \frac{\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}}{4}$

Solution of a system of differential equation:-

A system of differential equation.

$$\dot{x}_1 = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n + b_1 u_1$$

$$\dot{x}_2 = a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} \cancel{x_n} + b_2 u_1$$

$$\vdots$$

$$\dot{x}_n = a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n + b_n u_1$$

Can be written in a matrix form as :-

$$\dot{x} = A x + B u$$

where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = [\, b_1 \; b_2 \; \cdots \; b_n \,]$

$$\dot{x} = a x + b u$$

the solution of homogenous is

$$x_h(t) = e^{at} x(0)$$

the solution of non-homogenous is

$$x_p(t) = \int_0^t e^{a(t-\tau)} * B * u(\tau) \cdot d\tau$$

$$\therefore \quad x(t) = x_h(t) + x_p(t)$$